

# Plasticity Constitutive Law for Sea Ice

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## LONG-TERM GOALS

My long term goal is to develop sea ice dynamics models that describe behavior on kilometer scales and larger, and to base these models on the smaller scale physical processes known to control leading, rafting, and ridging. I also aim to implement these models in the next-generation Polar Ice Prediction System (PIPS).

## OBJECTIVES

My contract has the following tasks: (1) complete development of the constitutive law and test it with  $0d$  calculations, (2) incorporate the model into a  $2d$  numerical code and test its performance by comparing behavior with observed behavior, (3) extend my ambient noise model by relating the noise-generating model to ice mechanical behavior, (4) help transfer the new models into fleet systems such as PIPS, and (5) publish the results. The contract was expanded during 1999 to accelerate development of the ice dynamics model for PIPS 3.0. My tasks include adaptation of an implicit finite element method and reformulation of the redistribution function in the oriented thickness distribution description.

## APPROACH

We have developed a new anisotropic plasticity constitutive law to describe and forecast ice stress, deformation, lead direction, and ice condition at scales from a few kilometers to hundreds of kilometers [Coon, *et al.*, 1992, 1998; Pritchard, 1998a]. The essential difference between this anisotropic plasticity model and previous isotropic models is that it can describe the formation and direction of each new lead or ridge system and track its thickness distribution. I am working with M. Coon to develop this constitutive law.

Ice conditions are described by an oriented thickness distribution  $g(h, \theta)$  that describes the fraction of ice that has thickness  $h$  and orientation  $\theta$  [Pritchard, 1998a]. This description is needed for Eulerian simulations, where we must advect complete fields through the grid.

I have been reviewing numerical methods used in the finite element codes to determine how to adapt an implicit scheme to the new ice dynamics model. Several methods are attractive. They offer us the opportunity to use time steps representative of the physical processes, perhaps on the order of a day. The algorithms are more complicated, and each step is slower than the existing explicit leap frog scheme. The new methods will be tested in a finite element code that I developed earlier for ice

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dynamics simulations. I am talking with Dr. H. L. Schreyer of the University of New Mexico to learn how the material science community has adapted these methods to their problems.

I have been working with R. Bourke, J. Wilson, and W. D. Hibler III to modify the PIPS 2 ice dynamics model so that it can serve as an acoustic source model. My experience using energy dissipation as a proxy variable to describe noise generated by ridging [Pritchard, 1984, 1990, 1993] provides this team with the needed background. These results will provide guidance to development of an acoustical model for PIPS 3.0.

## WORK COMPLETED

The anisotropic elastic plastic constitutive law and the oriented thickness distribution have been formulated [Coon, *et al.*, 1992, 1998; Pritchard, 1998a]. I have developed a code to integrate the anisotropic plasticity constitutive law for given deformation histories [Pritchard, 1998b]. This *0d* model, which is similar to the column models used in large-scale climate simulations will be used to define material properties, in preparation for full *2d* simulations.

The redistribution function has been restudied, and a new formulation developed. I have extended the formulation presented by Pritchard [1998a] to correct the behavior under all ice conditions. My earlier function was not adequate when thin ice in a lead was lacking.

I participated in a PIPS 3.0 Development Team meeting at the National Ice Center. This group discussed concepts for the next-generation ice dynamics model, and the anisotropic plasticity law will be used in PIPS 3.0.

A finite element numerical method has been identified as a candidate for integrating the next-generation anisotropic plasticity model.

## RESULTS

Redistribution Function. At each instant and location, the plasticity law determines the stress state and the plastic stretching  $\mathbf{D}_p$  (the importance of this fact has not always been recognized). The stress state at plastic failure determines the orientation of ice that is deforming. There can be no stretching along a lead or ridge unless the surrounding thicker ice is failing. If the plastic stretching is composed of contributions from more than one flow rule, then ice may be failing in multiple orientations. This may happen when the stress is on a corner of the yield surface, or it is on an isotropic part of the yield surface.

The plastic stretching is then written as

$$\mathbf{D}_p = \sum_j \mathbf{P}_j$$

where

$$\mathbf{P}_j = \ddot{\epsilon}_j \frac{\partial \ddot{\epsilon}_j}{\partial \dot{\phi}}$$

is the contribution to plastic stretching from the  $\theta_j$  orientation.

If the plastic stretching is opening, then open water forms in that orientation (or orientations). If the plastic stretching is closing, thin ice in the lead having that orientation will ridge, or if all of this thin ice has been consumed, then ice having all orientations (isotropic ice) will ridge. This is the fundamental extension to the redistribution function introduced by *Pritchard* [1998a]. When isotropic ice is being deformed, the behavior can follow the isotropic thickness distribution theory introduced by *Thorndike, et al.* [1975]. For the oriented thickness distribution  $g(h, \hat{e})$ , the isotropic fractions are defined by

$$i(h) = \min_{\hat{e}} g(h, \hat{e}) .$$

We assume that only the thinner (or thinnest) fractions participate in the ridging. A manuscript describing the new redistribution function is now being prepared.

For each deforming orientation, the plastic stretching should be expressed in lead coordinates in that orientation. This is a simple, but conceptually necessary, step. As a result, the model conforms to the methods used by M. Coon to analyze SAR imagery. He has shown that only lead coordinates allow the areas to be calculated accurately. Of course, since stretching along a lead must be zero and the trace operator is invariant under coordinate rotation, the rate of opening or closing across a lead  $d_j$  must equal the rate of dilating  $tr \mathbf{P}_j$ , expressed in either lead or global coordinates.

Numerical Method. The momentum balance equation describes the sum of inertial, Coriolis, and sea surface tilt accelerations that result from the applied air and water stresses and divergence of the ice stress resultant.

$$m(\ddot{\mathbf{U}} + f\mathbf{k} \times \dot{\mathbf{U}} - g\nabla\eta) = \hat{\sigma}_a - \hat{\sigma}_w + \nabla \cdot \boldsymbol{\sigma}$$

where  $\mathbf{U}$  is displacement,  $\dot{\mathbf{U}}$  is velocity, and  $\ddot{\mathbf{U}}$  is acceleration. The elastic plastic constitutive law is

$$\dot{\boldsymbol{\sigma}} = \mathbf{M}_{ep} \dot{\mathbf{e}}$$

where  $\mathbf{M}_{ep}$  is the elastic plastic modulus tensor. This modulus tensor results from combining the linear elastic closure, the kinematic relationship, the flow rule, and yield surface. An estimate of the stress state at time  $t+\theta\Delta t$  comes from a difference approximation to this constitutive law.

The Wilson  $\theta$ -method was developed as an implicit finite element integration scheme for dynamic structural mechanics problems. It may be derived by assuming that the acceleration is linear over the time period  $(t, t + \theta\Delta t)$

$${}^{t+\tau}\ddot{\mathbf{U}} - {}^t\ddot{\mathbf{U}} = \frac{\tau}{\theta\Delta t} ({}^{t+\theta\Delta t}\ddot{\mathbf{U}} - {}^t\ddot{\mathbf{U}})$$

Integrating the linear acceleration provides the velocity and displacement variable as a function of initial values and the acceleration. Velocity varies as a quadratic function of time

$${}^{t+\tau}\dot{\mathbf{U}} = {}^t\dot{\mathbf{U}} + {}^t\ddot{\mathbf{U}}\tau + \frac{\tau^2}{2\theta\Delta t} ({}^{t+\theta\Delta t}\ddot{\mathbf{U}} - {}^t\ddot{\mathbf{U}})$$

and displacement varies as a cubic function of time

$${}^{t+\tau}\mathbf{U} = {}^t\mathbf{U} + {}^t\ddot{\mathbf{U}}\tau + \frac{1}{2}{}^t\ddot{\mathbf{U}}\tau^2 + \frac{\tau^3}{6\theta\Delta t}({}^{t+\theta\Delta t}\ddot{\mathbf{U}} - {}^t\ddot{\mathbf{U}})$$

The velocity at time  $t + \theta\Delta t$  is

$${}^{t+\theta\Delta t}\dot{\mathbf{U}} = {}^t\dot{\mathbf{U}} + {}^t\ddot{\mathbf{U}}\theta\Delta t + \frac{\theta\Delta t}{2}({}^{t+\theta\Delta t}\ddot{\mathbf{U}} + {}^t\ddot{\mathbf{U}})$$

and the displacement is

$${}^{t+\theta\Delta t}\mathbf{U} = {}^t\mathbf{U} + {}^t\ddot{\mathbf{U}}\theta\Delta t + \frac{1}{2}{}^t\ddot{\mathbf{U}}\theta^2\Delta t^2 + \frac{\theta^2\Delta t^2}{6}({}^{t+\theta\Delta t}\ddot{\mathbf{U}} - {}^t\ddot{\mathbf{U}})$$

Finally, we solve for acceleration and velocity in terms of the displacement (all at time  $t + \theta\Delta t$ ) and known values at time  $t$ , we find that

$${}^{t+\theta\Delta t}\ddot{\mathbf{U}} = \frac{6}{\theta^2\Delta t^2}({}^{t+\theta\Delta t}\mathbf{U} - {}^t\mathbf{U}) - \frac{6}{\theta\Delta t}{}^t\dot{\mathbf{U}} - 2{}^t\ddot{\mathbf{U}}$$

and

$${}^{t+\theta\Delta t}\dot{\mathbf{U}} = \frac{3}{\theta\Delta t}({}^{t+\theta\Delta t}\mathbf{U} - {}^t\mathbf{U}) - 2{}^t\dot{\mathbf{U}} - \frac{\theta\Delta t}{2}{}^t\ddot{\mathbf{U}}.$$

When the acceleration and velocity at time  $t + \theta\Delta t$  are expressed in terms of the displacement, the momentum balance equation at time  $t + \theta\Delta t$  therefore depends only on the displacement (and the stress state)

Momentum balance in finite element method notation is

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = \mathbf{R}$$

where  $\mathbf{M}$  is the generalized mass matrix,  $\mathbf{C}$  is the generalized damping matrix, and  $\mathbf{K}$  is the generalized stiffness matrix. The load vector on the right hand side  $\mathbf{R}$  must be similarly evaluated at this time. The displacement at time  $t + \theta\Delta t$  can be determined. In practice, we can reformulate the difference equations in terms of relative displacement over the time step  $\theta\Delta t$ . This is an effective approach that was used by *Pritchard, et al.* [1983]

If the parameter  $\theta$  exceeds 1.37, the scheme is unconditionally stable. This property is important for the PIPS 3.0 ice dynamics model. I envisage a 3-hour time step, which could describe all essential processes including tidal and inertial oscillations. This is a practicable scheme where we must solve the momentum balance equation for the displacement at the advanced time. The method is therefore implicit because the stress state must be calculated at the same time.

## IMPACT/APPLICATION

Treating sea ice as an anisotropic material provides explicit information on whether or not leads exist in a region, and their orientations. Knowing when and where leads will appear is useful to Navy operations. Accounting for leads (and ridges) explicitly, will allow us to describe ice behavior more

accurately. Perhaps the improved accuracy will be most noticeable in lower resolution models because the discontinuous behavior will be better described within each grid cell.

## TRANSITIONS

The constitutive law is available for use in other research codes and in Navy operational ice forecasting systems. I will work with Navy personnel and contractors to convert the research code into coding that is compatible with existing codes such as PIPS. The National Ice Center has identified this new anisotropic constitutive law as the leading candidate for including in PIPS 3.0.

## RELATED PROJECTS

The technical approach requires collaboration with M. Coon (NorthWest Research Associates, Inc.). A formal Management Plan was prepared to describe how we will collaborate, and how we will prevent duplication of our efforts so that we meet our goals with the scarce resources available.

I am working with R. Bourke, J. Wilson, and W. Hibler to install an ambient noise model into future versions of PIPS. I am helping to define the best proxy variables available in the PIPS 2 model.

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- Coon, M. D., Knoke, G. S., and Echert, D. C., and Pritchard, R. S. • An Oriented Thickness Distribution for Sea Ice," submitted.